Control Problems with State Constraints for the Penrose-Fife Phase-field model

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Abstract: This article gives an optimality system for a control problem with state constraints for a Penrose-Fife model for phase transitions.

Key-words: state constraints, optimality system, phase transition.

(Résumé : tsvp)

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Problèmes de contrôle avec contraintes sur l'état pour le modèle de Penrose-Fife.

Résumé : On étudie des problèmes de contrôle avec des contraintes sur l'état pour des modéles non linéaires de type Penrose-Fife "Phase-field". On dérive des conditions d'optimalité pour les problèmes de contrôle.

Mots-clé : contraintes sur l'état, condition d'optimalité, transition de phase

1 Introduction

In this article we consider optimal control problems governed by the following system of quasi-linear parabolic equations,

$$\phi_t = K_1 \Delta\phi - s_0'(\phi) - \frac{\lambda(\phi)}{T} \tag{1}$$

$$T_t = -M_1 \Delta\left(\frac{1}{T}\right) - \lambda(\phi)\phi_t + v, \tag{2}$$

on $Q = \Omega \times (0, t^*)$, where $\Omega \subset \mathbb{R}^3$ is a bounded domain with a sufficiently smooth boundary $\partial\Omega$. s_0 denotes a double well potential. We let $\partial Q = \partial\Omega \times (0, t^*)$, and impose the following boundary conditions

$$\frac{\partial T}{\partial n} = -\alpha\,(T - w) \qquad \text{on} \qquad \partial Q \tag{3}$$

$$\frac{\partial \phi}{\partial n} = 0 \qquad \text{on} \qquad \partial Q, \tag{4}$$

and the initial conditions

$$\phi(x,0) = \phi_0(x), \quad T(x,0) = T_0(x). \tag{5}$$

These equations arise in a model for phase transitions introduced by Penrose and Fife [10]. In this setting T denotes the absolute temperature, and ϕ a non-conserved order-parameter.

Several papers have appeared in connection with the existence and uniqueness of solutions to this system as well as other analytical aspects of this system and related systems. We refer the reader to [7, 13, 4, 6, 8, 9] for some specific treatments. A more general discussion of systems of this type can be found in [2].

We will make similar assumptions in this article as in [7, 13], namely, for the potential s_0 we will assume that either

- **(A)** $s_0 \in C^3(\mathbb{R})$ and there exists a constant $C > 0$ such that $s_0''(\phi) > -C$ for all $\phi \in \mathbb{R}$.

  or

- **(B)** $s_0 = \phi \log\phi + (1-\phi)\log(1-\phi)$.

Furthermore, we will make the following simplifying assumptions:

- $\lambda(\phi) = a\phi + b$, for a positive constant a. To simplify notations we will, without loss of generality, use $a = 1$ and $b = 0$, i.e. use $\lambda(\phi) = \phi$.
- In the boundary conditions we let $\alpha = 1$.

To state an existence result we have to make some regularity assumptions and compatibility conditions. In particular we have

(H1) $\phi_0 \in H^4(\Omega); \frac{\partial \phi}{\partial n}(x) = 0, \forall x \in \partial\Omega; \frac{\partial}{\partial n}\left(-s_0'(\phi_0) + \frac{\phi_0}{T_0} + \Delta\phi_0\right)(x) = 0, \forall x \in \partial\Omega.$

(H2) $T_0 \in H^3(\Omega); \tilde{T}(x) = \frac{\partial T_0}{\partial n}(x) + T_0(x) > 0, \forall x \in \partial\Omega; T_0(x) > 0, \forall x \in \overline{\Omega}.$

Finally, we introduce some Banach spaces which will be widely used throughout this article.

$$
\begin{aligned}
X_1 &= C([0,t^*]; H^4(\Omega)) \cap C^1([0,t^*]; H^2(\Omega)) \cap C^2([0,t^*]; L^2(\Omega)),\\
X_2 &= C([0,t^*]; H^3(\Omega)) \cap C^1([0,t^*]; H^1(\Omega)) \cap H^{4,2}(Q),\\
V &= H^2(0,t^*; L^2(\Omega)) \cap H^1(0,t^*; H^2(\Omega)),\\
W &= H^2(0,t^*; H^{\frac{3}{2}}(\partial\Omega)).
\end{aligned}
$$

Using these conditions one can prove the following existence result (cf. [7, 13])

Proposition 1 *Suppose (H1) and (H2) are satisfied. Then there exists a unique global smooth solution $(\phi, T) \in X_1 \times X_2$ to the equations (1)–(2) with the boundary conditions (3)–(4). Furthermore, there exists a constant $c_{t^*} > 0$ such that $T(x,t) \geq c_{t^*}$ for all $(x,t) \in \overline{Q}$, and in the case* **(B)** *there exist constants $0 < a_{t^*} < b_{t^*} < 1$, such that $a_{t^*} \leq \phi(x,t) \leq b_{t^*}$ for all $(x,t) \in \overline{Q}$.*

In Section 2 of this article we will state the optimal control problem with state constrains and discuss it. In Section 3 we will investigate the related observation operator and prove its differentiability in the setting of Section 2. Finally, we will derive the necessary conditions for optimality in Section 4 of this paper.

2 Optimal Control Problem

The state equations (1)–(2) give rise to several interesting optimal control problems. In this article we want to control the state (ϕ, T) by using the source term v in (2) and the boundary term w in (4) as controls. However, we want to put local constraints on the state as well.

In order to formulate this problem in a precise manner we need to introduce some additional notation. We start by defining the cost functional

$$\begin{aligned} I(\phi, T; v, w) &= \frac{\alpha_1}{2}\left\|\phi(t^*) - \hat{\phi}(t^*)\right\|^2_{L^2(\Omega)} + \frac{\alpha_2}{2}\left\|T - \hat{T}\right\|^2_{L^2(Q)} \\ &\quad + \frac{\alpha_3}{2}\|v\|^2_{L^2(Q)} + \frac{\alpha_4}{2}\int_0^{t^*}\|w(t)\|^2_{L^2(\partial\Omega)}\,dt, \end{aligned} \tag{6}$$

for given target functions $\hat{\phi} \in X_1$ and $\hat{T} \in X_2$. Next let

$$\begin{aligned} \tilde{W} = \Big\{ w \in W : \quad & w(x,0) = \tilde{T}(x), \quad \forall x \in \partial\Omega; \\ & w(x,t) \geq \beta, \quad |w_t(x,t)| < k, \quad \forall (x,t) \in \partial Q \Big\}, \end{aligned}$$

where $\tilde{T}$ is the function introduced by (H2) and β and k are suitably chosen positive constants. We use this set to introduce

$$K = V \times \tilde{W}$$

The set $\mathcal{U}_{\rm ad}$ of admissible controls is a closed, convex and bounded subset of K.

To state the local state constraints we use the constants $0 < K_1 < K_2$ and $K_3 < K_4$ to define

$$\mathcal{Y}_{\rm ad} = \left\{ (\phi, T) \in X_1 \times X_2 : K_1 \leq T(x,t) \leq K_2 \wedge K_3 \leq \phi(x,t) \leq K_4, \forall (x,t) \in Q \right\}, \tag{7}$$

the set of admissible states. Note, that this set has a nonempty interior.

We can now state the optimal control problem under consideration.

Optimal Control Problem (CP)

Minimize $I(\phi, T; v, w)$ under the following conditions:

1. (ϕ, T) satisfies the state equations (1)–(2) and the initial and boundary conditions (3)–(5).
2. $(v, w) \in \mathcal{U}_{\rm ad}$.
3. $(\phi, T) \in \mathcal{Y}_{\rm ad}$.

Remarks:

- Clearly the initial values (ϕ_0, T_0) must also satisfy the constraints $K_1 \leq T(x) \leq K_2$ and $K_3 \leq \phi_0(x) \leq K_4$ for all $x \in \Omega$.
- The authors of [14] considered a similar but weaker control problem. In particular, they did not put local constraints on the state. Moreover, their treatment focuses on the function $s_0'(\phi) = \phi - \phi^3$. However, this latter restriction can be easily removed and their argument extended to the cases (A) and (B) investigated here (see [5], for a sketch of this argument). We can therefore use their results whenever they are applicable.

In the study of the control problem (CP) it is useful to introduce the observation operator S. To do this we define the space of observations B by

$$B = (C([0,t^*]; H^2(\Omega))) \times (C([0,t^*]; H^2(\Omega))). \tag{8}$$

Next define

$$\begin{aligned} S &: K \to B \qquad (9)\\ S &: (v,w) \mapsto (\phi, T), \qquad (10)\end{aligned}$$

i. e. S assigns to every pair $(v,w) \in K$ the pair (ϕ, T) which solves (1)–(5) for the given v and w. Since $X_1 \times X_2 \subset B$ and by virtue of Proposition 1 this operator S is well defined. Using this operator one sees that the cost functional $I(\phi, T; v, w)$ depends only on the controls v and w i.e. we can rewrite it as

$$J(v,w) = I(\phi, T; v, w)|_{(\phi,T)=S(v,w)}\,.$$

In the following section we will study the properties of this operator S. In Section 4 these properties will be used to derive the necessary conditions of optimality.

Remark: Since the authors of [14] did not consider state constraints, they could use a larger space of observations with a coarser topology.

3 Differentiability of the Observation Operator

We now turn our attention to the observation operator S defined in (9)–(10). This operator is well-defined, and – also due to Proposition 1 – there exist positive constants α and γ such that

$$\begin{aligned} \|\phi\|_{X_1} + \|T\|_{X_2} &\leq \alpha, \quad \forall (v,w) \in \mathcal{U}_{\rm ad}, \qquad (11)\\ T(x,t) \geq \gamma &> 0, \quad \forall (x,t) \in \overline{Q}. \qquad (12)\end{aligned}$$

Moreover, if $s_0(\phi)$ is of the form given in case **B**, there exist constants $0 < \hat{a}_{t^*} < \hat{b}_{t^*} < 1$ such that

$$\hat{a}_{t^*} \leq \phi(x,t) \leq \hat{b}_{t^*}, \quad \forall (x,t) \in \overline{Q}. \tag{13}$$

In order to prove differentiability of the observation operator S one has to first improve the stability result of [14]. To do this we let $(\phi_i, T_i) = S(v_i, w_i), \quad i = 1,2$ and $(v_i, w_i) \in \mathcal{U}_{\mathrm{ad}}$. We define $\overline{\phi} = \phi_1 - \phi_2$, $\overline{T} = T_1 - T_2$, $\overline{v} = v_1 - v_2$, and $\overline{w} = w_1 - w_2$. Using these notations we have the following result.

Proposition 2 *There exists a constant $C > 0$ such that*

$$\begin{aligned}\max_{0<t<t^*} \left(\left\|\overline{\phi}_t(t)\right\|_{H^1}^2 + \left\|\overline{\phi}(t)\right\|_{H^3}^2 + \left\|\overline{T}\right\|_{H^2}^2 + \left\|\overline{T}_t(t)\right\|_{L^2}^2 \right) \quad &+ \quad \int_0^{t^*} \left\|\overline{\phi}_{tt}(t)\right\|^2 dt \\ + \int_0^{t^*} \left(\left\|\overline{\phi}_t(t)\right\|_{H^1}^2 + \left\|\overline{T}_t(t)\right\|_{H^1}^2 \right) dt \quad &\leq \quad C\overline{G}(\overline{v}, \overline{w}),\end{aligned} \tag{14}$$

where

$$\begin{aligned}\overline{G}(\overline{v}, \overline{w}) = \int_0^{t^*} &\left(\|\overline{w}_t(t)\|_{L^2(\partial\Omega)}^2 + \|\overline{v}_t(t)\|^2 + \|\overline{v}(t)\|^2 \right) dt \\ &+ \|\overline{v}(0)\|^2 + \|\overline{w}\|_{H^1(0,t^*;L^2(\partial\Omega))}^2 + \max_{0\leq t\leq t^*} \|\overline{w}(t)\|_{H^{\frac{1}{2}}(\partial\Omega)}^2 .\end{aligned} \tag{15}$$

Proof: From Theorem 2.1 of [14] we know that there exists a constant $\hat{C} > 0$ such that

$$\begin{aligned}\max_{0<t<t^*} \left(\left\|\overline{\phi}_t(t)\right\|_{H^1}^2 + \left\|\overline{\phi}(t)\right\|_{H^3}^2 + \left\|\overline{T}\right\|_{H^1}^2 \right) \quad &+ \quad \int_0^{t^*} \left(\left\|\overline{\phi}_{tt}(t)\right\|^2 + \left\|\overline{T}_t(t)\right\|^2 \right) dt \\ + \int_0^{t^*} \left(\left\|\overline{\phi}_t(t)\right\|_{H^1}^2 + \left\|\overline{T}(t)\right\|_{H^2}^2 \right) dt \quad &\leq \quad \hat{C} G(\overline{v}, \overline{w}),\end{aligned} \tag{16}$$

where

$$G(\overline{v}, \overline{w}) = \int_0^{t^*} \|\overline{v}(t)\|^2 \, dt + \|\overline{w}\|_{H^1(0,t^*;L^2(\partial\Omega))}^2 . \tag{17}$$

As in that paper $\overline{T}$ satisfies the following linear parabolic boundary value problem.

$$\overline{T}_t - \Delta\left(\overline{T}\zeta\right) \quad = \quad \phi_{1,t}\overline{\phi} - \phi_2\overline{\phi}_t + \overline{v}, \tag{18}$$

$$\left.\frac{\partial \overline{T}}{\partial n} + \overline{T}\right|_{\partial\Omega} = \overline{w}|_{\partial\Omega}\,, \qquad \overline{T}(x,0) = 0, \quad \forall x \in \overline{\Omega},. \tag{19}$$

where $\zeta = (T_1 T_2)^{-1}$. Observe that we have $\zeta \in L^\infty(Q)$ and $\nabla \zeta_t \in L^2(Q)$, because of the regularity properties of T_i from the existence and uniqueness results (cf. [7, 13]). We can now take the time derivative of (18) and (19) to obtain

$$\begin{aligned} \overline{T}_{tt} - \Delta\left(\overline{T}\zeta\right)_t &= \phi_{1,tt}\overline{\phi} - \phi_2\overline{\phi}_{tt} + \overline{\phi}_t^2 + \overline{v}_t, && (20)\\ &= f, && (21)\\ \left.\frac{\partial \overline{T}_t}{\partial n} + \overline{T}_t\right|_{\partial\Omega} &= \overline{w}_t|_{\partial\Omega}\,,. && (22) \end{aligned}$$

For the initial values of $\overline{T}_t$ observe

$$\begin{aligned} \overline{T}_t(x,0) &= \left(\Delta\left(\overline{T}\zeta\right) + \phi_{1,t}\overline{\phi} + \phi_2\overline{\phi}_t\right)(x,0) + \overline{v}(x,0)\\ &= \overline{v}(x,0). \end{aligned}$$

Furthermore, we observe that

$$\int_0^{t^*} \|f(t)\|^2\, dt \le c_1 G(\overline{v},\overline{w}) + \int_0^{t^*} \|\overline{v}_t(t)\|^2\, dt, \tag{23}$$

by the previous results. To continue with our proof we multiply (20) by $\overline{T}_t$ and integrate the resulting equation over Ω to arrive at

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\left\|\overline{T}_t(t)\right\|^2 + \int_\Omega \nabla\overline{T}_t(t)\cdot\nabla\left(\zeta(t)\overline{T}(t)\right)_t dx & - \int_{\partial\Omega} \overline{T}_t(t)\frac{\partial\left(\zeta(t)\overline{T}(t)\right)_t}{\partial n}\, dx && (24)\\ &\le \frac{\delta_1}{2}\|f(t)\|^2 + \frac{1}{2\delta_1}\left\|\overline{T}_t(t)\right\|^2 \end{aligned}$$

after applying (23) and Young's inequality. The value of δ_1 will be determined later. Next we observe that

$$\begin{aligned} \int_\Omega \nabla\overline{T}_t(t)\cdot\nabla\left(\zeta(t)\overline{T}(t)\right)_t dx &= \int_\Omega \nabla\overline{T}_t(t)\cdot\nabla\left(\zeta_t(t)\overline{T}(t) + \zeta(t)\overline{T}_t(t)\right) dx\\ &= \int_\Omega \zeta(t)\left|\nabla\overline{T}_t(t)\right|^2 dx + \int_\Omega \overline{T}(t)\nabla\overline{T}_t(t)\cdot\nabla\zeta_t(t)\, dx\\ &\quad + \int_\Omega \zeta_t(t)\nabla\overline{T}_t(t)\cdot\nabla\overline{T}(t)\, dx\\ &\quad + \int_\Omega \overline{T}_t(t)\nabla\overline{T}(t)\cdot\nabla\zeta(t)\, dx\\ &= \int_\Omega \zeta(t)\left|\nabla\overline{T}_t(t)\right|^2 dx + I_1(t) + I_2(t) + I_3(t) \end{aligned}$$

We can estimate the terms on the right of this last inequality individually as follows.

$$
\begin{aligned}
|I_1(t)| &\leq \left\|\nabla\overline{T}_t(t)\right\| \left\|\overline{T}(t)\right\|_{L^4(\Omega)} \|\nabla\zeta_t(t)\|_{L^4(\Omega)} \\
&\leq \frac{\delta_2}{2}\left\|\nabla\overline{T}_t(t)\right\|^2 + \frac{1}{2\delta_2}\|\nabla\zeta_t(t)\|^2_{H^1(\Omega)}\left\|\overline{T}(t)\right\|^2_{H^1(\Omega)}, \\
|I_2(t)| &\leq \left\|\nabla\overline{T}_t(t)\right\| \left\|\nabla\overline{T}(t)\right\|_{L^4(\Omega)} \|\zeta_t(t)\|_{L^4(\Omega)} \\
&\leq \frac{\delta_3}{2}\left\|\nabla\overline{T}_t(t)\right\|^2 + \frac{1}{2\delta_3}\|\zeta_t(t)\|^2_{H^1(\Omega)}\left\|\overline{T}(t)\right\|^2_{H^2(\Omega)}, \\
|I_3(t)| &\leq \left\|\nabla\overline{T}_t(t)\right\| \left\|\overline{T}_t(t)\right\| \|\nabla\zeta(t)\|_{L^\infty(\Omega)} \\
&\leq \frac{\delta_4}{2}\left\|\nabla\overline{T}_t(t)\right\|^2 + \frac{1}{2\delta_4}\|\nabla\zeta(t)\|^2_{L^\infty(\Omega)}\left\|\overline{T}_t(t)\right\|^2.
\end{aligned}
$$

In each of these inequalities one can estimate the integral over t of the second term on the right via $G(\overline{v},\overline{w})$. The values for δ_i will be determined later. For the boundary term we observe that

$$
\begin{aligned}
-\int_{\partial\Omega}\overline{T}_t\frac{\partial}{\partial t}\left(\frac{\partial\left(\zeta\overline{T}\right)}{\partial n}\right)dx &= -\int_{\partial\Omega}\overline{T}_t\frac{\partial}{\partial t}\left(\zeta\left(\overline{w}-\overline{T}\right)\right)dx \\
&\quad -\int_{\partial\Omega}\overline{T}_t\frac{\partial}{\partial t}\left(\overline{T}\zeta^2\left(T_1\left(w_2-T_2\right)+T_2\left(w_1-T_1\right)\right)\right)dx \\
&= \int_{\partial\Omega}\zeta\overline{T}_t^2\,dx - \int_{\partial\Omega}\overline{T}_t\left(\overline{w}-\overline{T}\right)\zeta_t\,dx - \int_{\partial\Omega}\zeta\overline{T}_t\overline{w}_t\,dx \\
&\quad + \int_{\partial\Omega}\overline{T}_t^2\left(\zeta^2\left(T_1\left(w_2-T_2\right)+T_2\left(w_1-T_1\right)\right)\right)dx \\
&\quad + \int_{\partial\Omega}\overline{T}_t\overline{T}\left(\zeta^2\left(T_1\left(w_2-T_2\right)+T_2\left(w_1-T_1\right)\right)\right)_t dx \\
&= \int_{\partial\Omega}\zeta\overline{T}_t^2\,dx + J_1(t) + +J_3(t) + J_4(t).
\end{aligned}
$$

We can again estimate the terms individually as follows.

$$
\begin{aligned}
|J_1(t)| &\leq \frac{\delta_5}{2}\left\|\overline{T}_t(t)\right\|^2_{L^2(\partial\Omega)} + \frac{c_5}{2\delta_5}\left(\|\overline{w}(t)\|^2_{L^4(\partial\Omega)} + \left\|\overline{T}(t)\right\|^2_{L^4(\partial\Omega)}\right)\|\zeta_t\|^2_{L^4(\partial\Omega)}, \\
|J_2(t)| &\leq \frac{\delta_6}{2}\left\|\overline{T}_t(t)\right\|^2_{L^2(\partial\Omega)} + \frac{c_6}{2\delta_6}\|\overline{w}_t(t)\|^2_{L^2(\partial\Omega)}, \\
|J_3(t)| &\leq c_7\left\|\overline{T}_t(t)\right\|^2_{L^2(\partial\Omega)} \leq \frac{\delta_7}{2}\left\|\nabla\overline{T}_t(t)\right\|^2 + \hat{c}_7\left\|\overline{T}_t(t)\right\|^2,
\end{aligned}
$$

$$
\begin{aligned}
|J_4(t)| \quad \leq \quad & \frac{\delta_8}{2}\left\|\overline{T}_t(t)\right\|^2_{L^2(\partial\Omega)} \\
& +\frac{1}{2\delta_8}\left\|\overline{T}\right\|^2_{L^4(\partial\Omega)}\left\|\left(\zeta^2\left(T_1\left(w_2-T_2\right)+T_2\left(w_1-T_1\right)\right)\right)_t\right\|^2_{L^4(\partial\Omega)}.
\end{aligned}
$$

From the trace theorem and the Sobolev imbedding theorem (see, for example, [1] for the Sobolev theorem for fractional exponents) we have the following continuous imbeddings

$$
\left\{v : v = u|_{\partial\Omega}\, ; u \in H^1(\Omega)\right\} \hookrightarrow H^{\frac{1}{2}}(\partial\Omega) \hookrightarrow L^4(\partial\Omega). \tag{25}
$$

Using this we can bound the time integrals of the second terms on the right by $\overline{G}(\overline{v},\overline{w})$. After choosing the δ_i's sufficiently small we combine all the estimates to get after integration over t

$$
\begin{aligned}
\frac{1}{2}\left\|\overline{T}_t(t)\right\|^2 + \hat{c}\int_0^t \left\|\overline{T}_t(s)\right\|^2_{H^1(\Omega)} ds \quad & \leq \quad C_1\overline{G}(\overline{v},\overline{w}) + \frac{1}{2}\left\|\overline{T}_t(0)\right\|^2 \\
& \leq \quad C_2\overline{G}(\overline{v},\overline{w}).
\end{aligned}
$$

The result now immediately follows from elliptic regularity estimates.

In order to formulate the next result we introduce the sets

$$
K^{\pm}(v,w) = \left\{(h,k) \in V \times W : \exists\lambda > 0 \text{ such that } (v \pm \lambda h, w \pm \lambda k) \in \mathcal{U}_{\mathrm{ad}}\right\}, \tag{26}
$$

for $(v,w) \in \mathcal{U}_{\mathrm{ad}}$.

Proposition 3 *Suppose (H1) and (H2) hold and $(v,w) \in \mathcal{U}_{\mathrm{ad}}$. Then the observation operator*

$$
S : K \to B,
$$

has a directional derivative $(\psi,\theta) = D_{(h,k)}S(v,w)$ in the direction $K^+(h,k)$. Furthermore, at $S(v,w) = (\phi,T)$, this directional derivative $(\psi,\theta) \in X_1 \times X_2$ is the unique solution of the linear parabolic initial-boundary value problem

$$
\begin{aligned}
\psi_t - \Delta\psi \quad &= \quad \psi\left(\frac{1}{T} - s_0''(\phi)\right) - \frac{\phi}{T^2}\theta, \\
\theta_t - \Delta\left(\frac{\theta}{T^2}\right) \quad &= \quad (\phi\psi)_t + h, \\
\frac{\partial\psi}{\partial n} = 0, \qquad & \frac{\partial\theta}{\partial n} + \theta = k, \quad \text{on } \partial\Omega, \\
\psi(0,x) \quad &= \quad \theta(0,x) = 0, \quad \text{on } \overline{\Omega}
\end{aligned}
$$

The corresponding result holds for the directional derivative $D_{(-h,-k)}S(v,w)$ at (v,w) in direction $(h,k) \in K^-(v,w)$.

Proof: As in [14] we let

$$(\phi^\lambda, T^\lambda) = S(v + \lambda h, w + \lambda k).$$

Furthermore, we use the notation of the previous Proposition and let

$$\overline{\phi} = \phi^\lambda - \phi; \quad \overline{T} = T^\lambda - T; \quad \zeta = \frac{1}{TT^\lambda}.$$

Finally, define

$$p = \overline{\phi} - \lambda\psi; \quad q = \overline{T} - \lambda\theta.$$

It is clear that the linear parabolic system in the statement admits a unique solution $(\psi, \theta) \in X_1 \times X_2$. To continue suppose that $(h, k) \in K^+(v, w)$ and suppose that there is a $\overline{\lambda} > 0$ such that $(v + \lambda h, w + \lambda k) \in \mathcal{U}_{\mathrm{ad}}, \quad \forall \lambda \in (0, \overline{\lambda})$. We have to show

$$\|(p, q)\|_B = o(\lambda), \quad \text{as } \lambda \to 0^+. \tag{27}$$

Using our notation p and q observe the following system of linear parabolic boundary value problems.

$$p_t - \Delta p = s_0'(\phi) - s_0'(\phi^\lambda) - \lambda s_0''(\phi)\psi + \frac{p}{T} - \frac{\phi}{T^2}q + \frac{\phi}{T}\overline{T}^2\zeta - \overline{\phi}\overline{T}\zeta \tag{28}$$

$$q_t - \Delta\left(\frac{q}{T^2}\right) = \phi_t p + \phi p_t + \overline{\phi}\overline{\phi}_t - \Delta\left(\frac{\overline{T}^2\zeta}{T}\right) \tag{29}$$

$$\frac{\partial p}{\partial n} = 0; \qquad \frac{\partial q}{\partial n} + q = 0; \quad \text{on } \partial Q \tag{30}$$

$$0 = p(x, 0) = q(x, 0) \tag{31}$$

We prove (27) in several steps.
<u>Step1:</u>
In [14] the authors show that

$$\max_{0 \le t \le t^*}\left(\|p(t)\|_{H^1}^2 + \|q(t)\|^2\right) + \int_0^{t^*}\left(\|p_t(s)\|^2 + \|q(s)\|_{H^1}^2 + \|p(s)\|_{H^2}^2\right) ds \le C\lambda^4, \tag{32}$$

for a suitable constant $C > 0$. We continue from there by multiplying (29) by $\left(\frac{q}{T^2}\right)_t$. After integrating the resulting equation over $\Omega \times [0, t]$ we obtain

$$\int_0^t \left\|\frac{q_t}{T}\right\|^2 ds + \frac{1}{2}\left\|\nabla\left(\frac{q}{T^2}\right)(t)\right\|^2 - \int_0^t \int_{\partial\Omega} \left(\frac{q}{T^2}\right)_t \frac{\partial}{\partial n}\left(\frac{q}{T^2}\right) dx ds \tag{33}$$

$$= \int_0^t \int_\Omega f\left(\frac{q_t}{T^2} - 2\frac{qT_t}{T^3}\right) dx ds - 2\int_0^t \int_\Omega \frac{q_t q T_t}{T^3} dx ds, \tag{34}$$

where f is given by

$$\phi_t p + \phi p_t + \overline{\phi}\overline{\phi}_t - \Delta\left(\frac{\overline{T}^2\zeta}{T}\right).$$

From Proposition 1 and the earlier estimates we see that

$$\int_0^{t^*} \|f(s)\|^2 \, ds \leq C_1\lambda^4,$$

for a suitable constant C_1.Furthermore, we have

$$\int_0^{t^*} \left\|\frac{qT_t}{T^3}(s)\right\|^2 \, ds \leq C_2\lambda^4,$$

due to earlier estimates. For the boundary term we observe

$$\frac{\partial}{\partial n}\left(\frac{q}{T^2}\right) = \frac{q}{T^2}\left(\frac{w}{T} - 1\right).$$

Therefore we have

$$\begin{aligned}
\left|\int_0^t \int_{\partial\Omega} \left(\frac{q}{T^2}\right)_t \frac{\partial}{\partial n}\left(\frac{q}{T^2}\right) dxds\right| &= \left|\int_0^t \int_{\partial\Omega} \left(\frac{q}{T^2}\right)_t \frac{q}{T^2}\left(1 - \frac{w}{T}\right) dxds\right| \\
&\leq c_1 \left\|\frac{q}{T^2}(t)\right\|^2 \\
&\quad + c_2 \int_0^t \int_{\partial\Omega} q^2 \left|\left(\frac{w}{T}\right)_t\right| dxds \\
&\leq c_1\delta \left\|\nabla\left(\frac{q}{T^2}(t)\right)\right\|^2 + c_3 \|q(t)\|^2 \\
&\quad + c_2 \int_0^t \|q(s)\|^2_{L^4(\partial\Omega)} \left\|\left(\frac{w}{T}\right)_t\right\| ds \\
&\leq c_1\delta \left\|\nabla\left(\frac{q}{T^2}(t)\right)\right\|^2 + c_4\lambda^4 + c_5 \int_0^t \|q(s)\|^2_{H^1} \, ds
\end{aligned}$$

In the last line of this estimate we used (25). Combining these estimates, using Young's inequality and choosing δ sufficiently small we obtain

$$\max_{0\leq t\leq t^*} \left\|\nabla\left(\frac{q}{T^2}\right)(t)\right\|^2 + \int_0^{t^*} \left\|\frac{q_t}{T}\right\|^2 ds \leq C_3\lambda^4.$$

It immediately follows

$$\max_{0\le t\le t^*} \|q(t)\|^2_{H^1} + \int_0^{t^*} \|q_t\|^2 \, ds \le C_4\lambda^4. \tag{35}$$

Step 2:

In the next step, we take the derivative of (28) with respect to t to get

$$p_{tt} - \Delta p_t = \Big(s_0'(\phi) - s_0'(\phi^\lambda) - \lambda s_0''(\phi)\psi\Big)_t + \left(\frac{p}{T} - \frac{\phi}{T^2}q + \frac{\phi}{T}\overline{T}^2\zeta - \overline{\phi T}\zeta\right)_t. \tag{36}$$

We observe that

$$\begin{aligned} |F_{1,t}| &= \left|\Big(s_0'(\phi) - s_0'(\phi^\lambda) - \lambda s_0''(\phi)\psi\Big)_t\right| \\ &\le \left|\phi_t\left(s_0''(\phi) - s_0''(\phi^\lambda) - s_0'''(\phi)\overline{\phi}\right)\right| + |s_0'''(\phi)\phi_t p| \\ &\quad + |s_0''(\phi)p_t| + \left|\Big(s_0''(\phi^\lambda) - s_0''(\phi)\Big)\overline{\phi}\right|. \end{aligned}$$

Using the mean-value theorem one easily sees that

$$\int_0^{t^*} \|F_{1,t}(s)\|^2 \, ds \le c_6\lambda^4, \tag{37}$$

for a suitable constant c_6. Next we observe that

$$\begin{aligned} F_{2,t} &= \frac{p_t}{T} - \frac{pT_t}{T^2} - \frac{\phi_t}{T^2}q + 2\frac{\phi T_t}{T^3}q - \frac{\phi}{T^2}q_t + \frac{\phi_t}{T}\overline{T}^2\zeta - \frac{\phi}{T^2}T_t\overline{T}_2\zeta \\ &\quad + 2\frac{\phi}{T}\overline{TT}_t\zeta + \frac{\phi}{T}\overline{T}^2\zeta_t - \overline{\phi}_t\overline{T}\zeta - \overline{\phi T}_t\zeta - \overline{\phi T}\zeta_t \end{aligned}$$

Since both ϕ_t and T_t are elements of $C([0,t^*];H^1(\Omega))$ we see that

$$\int_0^{t^*} \|F_{2,t}(s)\|^2 \, ds \le c_7\lambda^4, \tag{38}$$

for a suitable constant c_7. So if one multiplies (36) by p_t and integrates the result over $\Omega\times[0,t]$ one gets immediately

$$\max_{0\le t\le t^*} \|p_t(t)\|^2 + \int_0^{t^*} \|p_t(s)\|^2_{H^1} \, ds \le C_5\lambda^4 \tag{39}$$

We can now apply the standard elliptic regularity estimates to get

$$\max_{0\le t\le t^*} \|p(t)\|^2_{H^2} \le C_6\lambda^4. \tag{40}$$

Furthermor, we can multiply (36) by p_{tt}, integrate the result over $\Omega \times [0,t]$ and use (37) and (38) again to get

$$\max_{0\le t\le t^*} \|p_t\|^2_{H^1} + \int_0^{t^*} \|p_{tt}(s)\|^2 \, ds \le C_7 \lambda^4, \tag{41}$$

for a suitable constant C_7.
Step 3:
To continue we take the time derivative of (29) to obtain

$$q_{tt} - \Delta \left(\frac{q}{T^2}\right)_t = F_{3,t}(x,t), \tag{42}$$

where

$$F_{3,t} = \left(\phi_t p + \phi p_t + \overline{\phi\phi}_t - \Delta\left(\frac{\overline{T}^2 \zeta}{T}\right)\right)_t .$$

To simplify notations we introduce $\hat\zeta = \frac{\zeta}{T}$, which has the same properties as ζ. We observe that

$$\begin{aligned}\Delta\left(\overline{T}^2 \hat\zeta\right)_t &= 2\overline{T}_t \hat\zeta \Delta \overline{T} + 4\hat\zeta \nabla \overline{T} \cdot \nabla \overline{T}_t + 4\overline{T}_t \nabla \overline{T} \cdot \nabla \hat\zeta + 2\overline{T}\hat\zeta \Delta \overline{T}_t + 4\overline{T}\nabla\overline{T}_t \cdot \nabla\hat\zeta \\ &\quad + 2\overline{T}\overline{T}_t \Delta\hat\zeta + 2\left|\nabla\overline{T}\right|^2 \hat\zeta_t + 2\hat\zeta_t \overline{T}\Delta\overline{T} + 4\overline{T}\nabla\overline{T}\cdot\nabla\hat\zeta_t + \overline{T}^2 \Delta\hat\zeta_t .\end{aligned}$$

Using the results of Proposition 1, we can bound $\left\|\overline{T}(t)\right\|_{H^2}$ by $c_9\lambda$ for a sufficiently large constant c_9. Furthermore, we know that $\overline{T}$ has the same regularity as $\hat\zeta$ which enables us to bound terms of the form

$$\int_0^{t^*} \left\|\overline{T}\right\|^2_{H^2} ds, \text{ and } \quad \max_{0\le t\le t^*} \left\|\overline{T}_t(t)\right\|_{H^1}$$

by constants. Combining these properties we see that

$$\int_0^{t^*} \left\|\Delta\left(\overline{T}^2\hat\zeta\right)_t (s)\right\|^2 ds \le c_{10}\lambda^2$$

for a suitable constant c_{10}. It follows that

$$\int_0^{t^*} \|F_{3,t}(s)\|^2 \, ds \le c_{11}\lambda^2. \tag{43}$$

We multiply (42) by q_t and integrate the result over $\Omega \times [0,t]$ to get

$$\begin{aligned}
\frac{1}{2}\|q_t(t)\|^2 \quad &+ \quad \int_0^{t^*}\int_\Omega \nabla q_t \cdot \nabla \left(\frac{q}{T^2}\right)_t dxds - \int_0^{t^*}\int_{\partial\Omega} q_t \frac{\partial}{\partial n}\left(\frac{q}{T^2}\right)_t dxds \\
&\leq \left(\int_0^{t^*} \|F_{3,t}(s)\|^2\, ds\right)^{\frac{1}{2}} \left(\int_0^{t^*} \|q_t(s)\|^2\, ds\right)^{\frac{1}{2}} \leq c_{12}\lambda^3,
\end{aligned}$$

for a suitable constant c_{12}. We next observe that

$$\begin{aligned}
\int_0^{t^*}\int_\Omega \nabla q_t \cdot \nabla\left(\frac{q}{T^2}\right)_t dxds \quad &= \quad \int_0^{t^*}\int_\Omega \nabla q_t \cdot \nabla\left(\frac{q_t}{T^2} - 2\frac{qT_t}{T^3}\right) dxds \\
&= \quad \int_0^{t^*} \left\|\frac{\nabla q_t}{T}(s)\right\|^2 ds \\
&\quad\; -2\int_0^{t^*}\int_\Omega \nabla q_t \cdot \left(\frac{q}{T^3}\nabla T + \frac{T_t}{T^3}\nabla q\right) dxds \\
&\quad\; +2\int_0^{t^*}\int_\Omega \nabla q_t \cdot \left(3\frac{qT_t}{T^4}\nabla T - \frac{q}{T^3}\nabla T_t\right) dxds
\end{aligned}$$

One sees that the mixed terms on the right can be treated via Young's inequality, and that we can use the fact that

$$\int_0^{t^*} \|q\|_{H^2}^2\, ds \leq c_{14}\lambda^4,$$

and the other earlier estimates on q. Finally we observe

$$\begin{aligned}
\int_0^{t^*}\int_{\partial\Omega} q_t \frac{\partial}{\partial n}\left(\frac{q}{T^2}\right)_t dxds \quad &= \quad \int_0^{t^*}\int_{\partial\Omega} q_t \left(\frac{1}{T^2}\frac{\partial q_t}{\partial n} - 2\frac{q_t}{T^3}\frac{\partial T}{\partial n}\right) dxds \\
&\quad\; +2\int_0^{t^*}\int_{\partial\Omega} q_t \left(3\frac{qT_t}{T^4}\frac{\partial T}{\partial n} - \frac{T_t}{T^3}\frac{\partial q}{\partial n} - \frac{q}{T^3}\frac{\partial T_t}{\partial n}\right) dxds \\
&= \quad -\int_0^{t^*}\int_{\partial\Omega} \frac{q_t^2}{T^2}\left(1 + 2\frac{1}{T}\frac{\partial T}{\partial n}\right) dxds \\
&\quad\; -2\int_0^{t^*}\int_{\partial\Omega} \frac{qq_t}{T^3}\left(T_t + \frac{\partial T_t}{\partial n} - 3\frac{T_t}{T^2}\frac{\partial T}{\partial n}\right) dxds
\end{aligned}$$

In the first term we observe that

$$1 + 2\frac{1}{T}\frac{\partial T}{T\partial n} \in L^\infty(\partial Q).$$

In the second term one has

$$\frac{1}{T}\left(T_t + \frac{\partial T_t}{\partial n} - 3\frac{T_t}{T^2}\frac{\partial T}{\partial n}\right) \in L^2(0,t^*;L^\infty(\partial\Omega)).$$

Using this we get

$$\left|2\int_0^{t^*}\int_{\partial\Omega}\frac{qq_t}{T^3}\left(T_t + \frac{\partial T_t}{\partial n} - 3\frac{T_t}{T^2}\frac{\partial T}{\partial n}\right)\,dxds\right| \le C_9\left(\int_0^{t^*}\left\|\frac{q_t}{T}\right\|^2_{L^2(\partial\Omega)}\|q\|^2_{L^2(\partial\Omega)}\,ds\right)^{\frac{1}{2}}.$$

Observe that

$$\|q(t)\|^2_{L^2(\partial\Omega)} \le C_{10}\lambda^4.$$

This implies that we are left to treat estimate of the form

$$\int_0^{t^*}\left\|\frac{q_t}{T}\right\|^2_{L^2(\partial\Omega)}\,ds.$$

We do this by using

$$\int_0^{t^*}\|g(s)\|^2_{L^2(\partial\Omega)}\,ds \le \delta\int_0^{t^*}\|\nabla g(s)\|^2\,ds + \hat{C}\int_0^{t^*}\|g(s)\|^2\,ds,$$

for a suitable constant $\hat{C}$. We can now combine all these estimates and use the properties of T to conclude

$$\max_{0\le t\le t^*}\|q_t(t)\| + \int_0^{t^*}\|\nabla q_t\|^2\,ds \le C_{11}\lambda^3, \tag{44}$$

for a suitable constant C_{11}. From elliptic regularity estimates it follows, that the same estimate holds for

$$\max_{0\le t\le t^*}\|q(t)\|^2_{H^2}.$$

This finishes the proof of the proposition.

4 Optimality conditions

We return to the optimal control problem (CP) of Section 2. We introduced the non-linear observation operator S (9)–(10). We can write S in components (S_1,S_2) as follows.

$$S(v,w) = \begin{pmatrix} S_1(v,w) \\ S_2(v,w)\end{pmatrix} = \begin{pmatrix}\phi \\ T\end{pmatrix} \tag{45}$$

Proposition 3 states that this operator is Gateaux differentiable with Gateaux derivative

$$DS(v,w)(h,k) = \begin{pmatrix} DS_1(v,w)(h,k) \\ DS_2(v,w)(h,k) \end{pmatrix} = \begin{pmatrix} \psi \\ \theta \end{pmatrix}, \tag{46}$$

given by the following system of linearized equations

$$\begin{aligned}
\psi_t - \Delta\psi &= \psi\left(\frac{1}{T} - s_0''(\phi)\right) - \frac{\phi}{T^2}\theta, && (47)\\
\theta_t - \Delta\left(\frac{\theta}{T^2}\right) &= (\phi\psi)_t + h, && (48)\\
\frac{\partial\psi}{\partial n} = 0, \qquad & \frac{\partial\theta}{\partial n} + \theta = k, \quad \text{on } \partial\Omega, && (49)\\
\psi(0,x) &= \theta(0,x) = 0, \quad \text{on } \overline{\Omega} && (50)
\end{aligned}$$

An application of the Lagrange multiplier rule implies that there exists $\lambda \geq 0$ and Borel measures $\mu_1, \mu_2, \mu_3, \mu_4$, with the properties:

$$\mu_i(\{(x,t) \in \overline{Q} | T(x,t) \neq K_i\}) = 0, \; i = 1,2, \tag{51}$$

$$\mu_i(\{(x,t) \in \overline{Q} | \phi(x,t) \neq K_i\}) = 0, \; i = 3,4, \tag{52}$$

such that

$$\lambda + |\mu_1| + |\mu_2| + |\mu_3| + |\mu_4| > 0,$$

where $|\mu_i|, i = 1, ..., 4$, denotes the norm of the measure μ_i.

The constants K_i are the ones given in the state constraints (7). To continue, we denote $\mu = \mu_1 - \mu_2, \nu = \mu_3 - \mu_4$.

The abstract optimality system for the control problem under consideration is given below by $(*)$ and $(**)$. The first condition takes the form

$$(*) \qquad\qquad \forall(\zeta,\eta) \in \mathcal{Y}_{ad} : \int(\eta - T)d\mu + \int(\zeta - \phi)d\nu \leq 0,$$

where $(\phi, T) = S(v,w)$ is a solution to the state equations for optimal controls $(v,w) \in \mathcal{U}_{\rm ad}$.

For the second condition,we recall the notation. We denote by $I(\phi, T; v, w)$ the cost functional, i.e. $J(v,w) = I(S_1(v,w), S_2(v,w); v, w)$, then the gradient of the cost functional, with respect to the controls takes the form

$$\begin{aligned}
\langle DJ(v,w),(h,k)\rangle &= \langle D_1I(\phi,T;v,w), D_1S(\phi,T)(h,k)\rangle \\
&+ \langle D_2I(\phi,T;v,w), D_2S(\phi,T)(h,k)\rangle \\
&+ \langle D_3I(\phi,T;v,w), h\rangle + \langle D_4I(\phi,T;v,w), k\rangle.
\end{aligned}$$

The second optimality condition is of the form

$$(**) \qquad \lambda\langle DJ(v,w),(h-v,k-w)\rangle + \langle [DS_2(v,w)]^*(h-v,k-w)],\mu\rangle$$
$$+\langle [DS_1(v,w)]^*(h-v,k-w)],\nu\rangle \geq 0,$$

for all $(h,k)\in\mathcal{U}_{\mathrm{ad}}$, where $[DS_i(v,w)]^*$ denotes the adjoint to $[DS_i(v,w)]$, $i=1,2$.

Assuming that the Slater condition is satisfied, we can take $\lambda = 1$. In the present case the Slater condition (S) means, that there exists an optimal control $(h_0,k_0)\in\mathcal{U}_{\mathrm{ad}}$ such that for all $(x,t)\in\overline{Q}$,

$$K_1 < T(x,t) + [DS_2(v,w)(h_0-v,k_0-w)](x,t) < K_2$$
$$K_3 < \phi(x,t) + [DS_1(v,w)(h_0-v,k_0-w)](x,t) < K_4$$

Furthermore, an adjoint state is introduced in order to simplify the latter optimality condition. To this end, we rewrite the linearized equations in the form.

$$\mathcal{L}_{11}(\psi)+\mathcal{L}_{12}(\theta)=0, \tag{53}$$
$$\mathcal{L}_{21}(\psi)+\mathcal{L}_{22}(\theta)=h, \tag{54}$$

with nonhomogeneous boundary condition

$$\frac{\partial\theta}{\partial n}+\theta=k, \quad \text{on } \partial\Omega, \tag{55}$$

where we denote

$$\mathcal{L}_{11}(\psi)=\psi_t-\Delta\psi-\psi\left(\frac{1}{T}-s_0''(\phi)\right), \tag{56}$$
$$\mathcal{L}_{12}(\theta)=\frac{\phi}{T^2}\theta, \tag{57}$$
$$\mathcal{L}_{21}(\psi)=-(\phi\psi)_t, \tag{58}$$
$$\mathcal{L}_{22}(\theta)=\theta_t-\Delta\left(\frac{\theta}{T^2}\right). \tag{59}$$

Therefore, for any functions $(q,p)\in V\times V$ it follows that

$$(\mathcal{L}_{11}(\psi)+\mathcal{L}_{12}(\theta),q)_V=0,$$
$$(\mathcal{L}_{21}(\psi)+\mathcal{L}_{22}(\theta),p)_V=(h,p)_V,$$

and the latter term, by an application of the associated Green formula, can be written in the form

$$(\mathcal{L}_{22}(\theta),p)_V=\mathcal{A}(\theta,p)-\ell\left(\frac{\partial\theta}{\partial n}+\theta,p\right) \tag{60}$$

with an appropriate bilinear form $\mathcal{A}(\cdot,\cdot)$, and a boundary form $\ell(\cdot,\cdot)$, which will be specified below. In particular, for $\frac{\partial\theta}{\partial n}+\theta=0$ it follows that

$$(\mathcal{L}_{22}(\theta),p)_V=\mathcal{A}(\theta,p)\ .$$

Hence, the system becomes

$$\begin{aligned}&(\mathcal{L}_{11}(\psi)+\mathcal{L}_{12}(\theta),q)_V=0,\\&(\mathcal{L}_{21}(\psi),p)_V+\mathcal{A}(\theta,p)=(h,p)_V+\ell(k,p).\end{aligned}$$

In order to identify the boundary form $\ell(k,p)$, we need Green formulae for the subsequent terms in the scalar product of the space $L^2(Q)$ which are given below.

$$-\left(\Delta\left(\frac{\theta}{T^2}\right),\phi\right)_{L^2(Q)}=\left(\nabla\left(\frac{\theta}{T^2}\right),\nabla\phi\right)_{L^2(Q)}-\int_0^{t^*}\left(\frac{\partial}{\partial n}\left(\frac{\theta}{T^2}\right),\phi\right)_{L^2(\partial\Omega)}dt,$$

and, in view of the boundary conditions, it follows that

$$\begin{aligned}-&\left(\frac{\partial}{\partial n}\left(\frac{\theta}{T^2}\right),\phi\right)_{L^2(\partial\Omega)}\\&=\left((\theta-k)\frac{1}{T^2},\phi\right)_{L^2(\partial\Omega)}-\left(\theta\frac{\partial}{\partial n}\frac{1}{T^2},\phi\right)_{L^2(\partial\Omega)}\\&=\left(\theta\left(\frac{1}{T^2}-\frac{\partial}{\partial n}\frac{1}{T^2}\right),\phi\right)_{L^2(\partial\Omega)}-\left(k\frac{1}{T^2},\phi\right)_{L^2(\partial\Omega)}.\end{aligned}$$

Similarly,

$$\begin{aligned}-\left(\nabla\left(\Delta\left(\frac{\theta}{T^2}\right)\right),\nabla(\phi)\right)_{L^2(\Omega)}&=\left(\Delta\left(\frac{\theta}{T^2}\right),\Delta\phi\right)_{L^2(\Omega)}\\&\quad-\left(\Delta\left(\frac{\theta}{T^2}\right),\frac{\partial\phi}{\partial n}\right)_{L^2(\partial\Omega)},\end{aligned}$$

and

$$\begin{aligned}-\left(\Delta\left(\Delta\left(\frac{\theta}{T^2}\right)\right),\Delta\phi\right)_{L^2(\Omega)}&=\left(\nabla\left(\Delta\left(\frac{\theta}{T^2}\right)\right),\nabla(\Delta\phi)\right)_{L^2(\Omega)}\\&\quad-\left(\frac{\partial}{\partial n}\Delta\left(\frac{\theta}{T^2}\right),\Delta\phi\right)_{L^2(\partial\Omega)}.\end{aligned}$$

We have also the following relation on the boundary $\partial\Omega$, (cf. [12]),

$$\Delta\left(\frac{\theta}{T^2}\right) = \Delta_\Gamma\left(\frac{\theta}{T^2}\right) + \kappa\frac{\partial}{\partial n}\left(\frac{\theta}{T^2}\right) + \frac{\partial^2}{\partial n^2}\left(\frac{\theta}{T^2}\right) \ ,$$

where Δ_Γ is the Laplace–Beltrami operator on $\Gamma = \partial\Omega$ and where κ denotes the tangential divergence of the normal vector field on Γ, i.e. $\kappa = \mathrm{div}_\Gamma n$, in the notation of [12].

The adjoint state equations are introduced in the following way. Assume that the functions $(q,p) \in V \times V$ satisfy the following variational equation

$$\begin{aligned}(\mathcal{L}_{11}(\zeta),q)_V \quad &+ \quad (\mathcal{L}_{12}(\eta),q)_V + (\mathcal{L}_{21}(\zeta),p)_V + (\mathcal{L}_{22}(\eta),p)_V \\ &= \quad \langle D_1I(\phi,T;v,w),\zeta\rangle + \int \zeta d\nu + \langle D_2I(\phi,T;v,w),\eta\rangle + \int \eta d\mu \end{aligned} \tag{61}$$

for all sufficiently smooth functions ζ,η satisfying homogeneous initial conditions and the homogeneous boundary conditions

$$\frac{\partial\zeta}{\partial n} = 0, \quad \text{and} \ \frac{\partial\eta}{\partial n} + \eta = 0. \tag{62}$$

Using the Lions projection theorem, see e.g. [15] for a variant of this theorem, one can show that these functions are uniquely determined.

The system (61) can be rewritten in the form

$$\begin{aligned}(\mathcal{L}_{11}(\zeta),q)_V \quad &+ \quad (\mathcal{L}_{12}(\eta),q)_V + (\mathcal{L}_{21}(\zeta),p)_V + \mathcal{A}(\eta,p) \\ &= \quad \langle D_1I(\phi,T;v,w),\zeta\rangle + \int \zeta d\nu + \langle D_2I(\phi,T;v,w),\eta\rangle + \int \eta d\mu,\end{aligned}$$

where the boundary condition $\frac{\partial\eta}{\partial n} + \eta = 0$ is imposed directly in the equation.

If we replace ζ,η by ψ,η, it follows that

$$\begin{aligned}\langle D_1I(\phi,T;v,w),\psi\rangle \quad &+ \quad \langle D_2I(\phi,T;v,w),\theta\rangle + \int \theta d\mu + \int \psi d\nu \\ &= \quad (\mathcal{L}_{11}(\psi),q)_V + (\mathcal{L}_{12}(\theta),q)_V + (\mathcal{L}_{21}(\psi),p)_V + \mathcal{A}(\theta,p) \\ &= \quad (\mathcal{L}_{11}(\psi),q)_V + (\mathcal{L}_{12}(\theta),q)_V \\ &\qquad + (\mathcal{L}_{21}(\psi) + \mathcal{L}_{22}(\theta),p)_V + \ell(k,p) \\ &= \quad (h,p)_V + \ell(k,p) \ .\end{aligned}$$

Using the above construction, it follows that for $\lambda = 1$ the necessary optimality conditions can be rewritten in the following form.

Theorem 1 *Assume that condition (S) is satisfied. Then there exist μ, ν and the adjoint state (q,p) such that the optimality system for the control problem includes the state equation, the adjoint state equation, and the condition $(*)$, as well as the following condition*

$$\begin{aligned}\langle D_3 I(\phi,T;v,w), h-v\rangle &+ (h-v,p)_V \\ &+ \langle D_4 I(\phi,T;v,w), k-w\rangle + \ell(k-w,p) \geq 0\end{aligned}$$

for all $(h,k) \in \mathcal{U}_{\mathrm{ad}}$.

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